# The Shapley Value

Game Theory Course:
Jackson, Leyton-Brown & Shoham

# Coalitional or Cooperative Games

- Question: what is a 'fair' way for a coalition to divide its payoff?
- This depends on how we define 'fairness.'
- One Approach: identify axioms that express properties of a fair payoff division.

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  - Cannot pay everyone their marginal Contribution!
- We will have to use some weighting system - how should it be designed?
- Shapley's axioms give us one answer...

# Symmetry

- $i$ and $j$ are interchangeable relative to $v$ if they always contribute the same amount to every coalition of the other agents.
  - for all $S$ that contains neither $i$ nor $j$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

**Axiom (Symmetry)**

*For any $v$, if $i$ and $j$ are interchangeable then $\psi_i(N, v) = \psi_j(N, v)$.*

- Interchangeable agents should receive the same shares/payments.

# Dummy Players

- $i$ is a dummy player if the amount that $i$ contributes to any coalition is 0.
  - for all $S$: $v(S \cup \{i\}) = v(S)$.

**Axiom (Dummy player)**

*For any $v$, if $i$ is a dummy player then $\psi_i(N, v) = 0$.*

- Dummy players should receive nothing.

# Additivity

- If we can separate a game into two parts $v = v_1 + v_2$, then we should be able to decompose the payments:

**Axiom (Additivity)**

*For any two $v_1$ and $v_2$, $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$ for each $i$, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition $S$.*

# Shapley Value

Given a coalitional game $(N, v)$, the Shapley Value divides payoffs among players according to:

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[ v(S \cup \{i\}) - v(S) \Big].$$

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**Theorem**

*Given a coalitional game $(N, v)$, there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy player and Additivity axioms: the Shapley Value*

# Understanding the Shapley Value

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[ v(S \cup \{i\}) - v(S) \Big].$$

This captures the "marginal contributions" of agent $i$, averaging over all the different sequences according to which the grand coalition could be built up.

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- For any such sequence, look at agent $i'$s marginal contribution when added: $[v(S \cup \{i\}) - v(S)]$.

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- Weight this quantity by the $|S|!$ ways the set $S$ could have been formed prior $i$'s addition and by the $(|N|-|S|-1)!$ ways the remaining players could be added.

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This captures the “marginal contributions” of agent $i$, averaging over all the different sequences according to which the grand coalition could be built up.

- Sum over all possible sets $S$ and average by dividing by $|N|!$: the number of possible orderings of all the agents.

# Understanding the Shapley Value:

# Two Partners Sharing their Profits:

$$v(\{1\}) = 1, v(\{2\}) = 2, v(\{1,2\}) = 4$$

# Shapley Value

- The Shapley Value allocates the value of a group according to marginal contribution calculations.

- Captured by some simple axioms and logic.

- Other axioms and approaches lead to other allocations of value - for example the "Core" up next.